

MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES OF HIGHER RANK ON PROJECTIVE SPACE \mathbb{P}^3

U. BRUZZO, D. MARKUSHEVICH, AND A. S. TIKHOMIROV

ABSTRACT. Symplectic instanton vector bundles on the projective space \mathbb{P}^3 constitute a natural generalization of mathematical instantons of rank 2. We study the moduli space $I_{n,r}$ of rank- $2r$ symplectic instanton vector bundles on \mathbb{P}^3 with $r \geq 2$ and second Chern class $n \geq r$, $n \equiv r \pmod{2}$. We give an explicit construction of an irreducible component $I_{n,r}^*$ of this space for each such value of n and show that $I_{n,r}^*$ has the expected dimension $4n(r+1) - r(2r+1)$.

1. INTRODUCTION

By a *symplectic instanton vector bundle* of rank $2r$ and charge n (shortly, a *symplectic (n, r) -instanton*) on the 3-dimensional projective space \mathbb{P}^3 we understand an algebraic vector bundle $E = E_{2r}$ of rank $2r$ on \mathbb{P}^3 with Chern classes

$$(1) \quad c_1(E) = 0,$$

$$(2) \quad c_2(E) = n, \quad n \geq 1,$$

supplied with a symplectic structure and satisfying the vanishing conditions

$$(3) \quad h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0.$$

By a symplectic structure we mean an anti-self-dual isomorphism

$$(4) \quad \phi: E \xrightarrow{\sim} E^\vee, \quad \phi^\vee = -\phi,$$

considered modulo proportionality. The vanishing of the first Chern class (1) follows from the existence of a symplectic structure (4), and if $r = 1$, then the two conditions are equivalent. We will denote the moduli space of symplectic (n, r) -instantons by $I_{n,r}$.

For $r = 1$ these bundles relate, via the so-called Atiyah-Ward correspondence, to rank-2 “physical” instantons over the 4-sphere S^4 , these being anti-self-dual connections with structure group $SU(2) = \mathbf{Sp}(1)$ [AW]. Important results on the moduli spaces $I_n = I_{n,1}$ of rank-2 instantons have been obtained recently: smoothness [JV] for all n , irreducibility [T] for odd n .

Much less is known about the moduli spaces $I_{n,r}$ for $r > 1$. In fact the symplectic instantons with $r > 1$ are as natural as those with $r = 1$, for they are related, via the same Atiyah-Ward correspondence, to the anti-self-dual connections over S^4 with structure group $\mathbf{Sp}(r)$, see [A]. As far as we know, the present paper is the first one addressing the properties of the corresponding spaces $I_{n,r}$. The tool we use to construct $I_{n,r}$ is the monad method; it originates in the work of Horrocks [H] and is known as the ADHM construction of instantons since [ADHM]. It was further sharpened in the work of Barth [B], Barth and Hulek [BH] and Tyurin [Tju1], [Tju2]. This method permits to encode the instantons, usual ones or symplectic of higher rank, by hyperwebs of quadrics.

For a sample of the physical literature about symplectic instantons, see e.g. [Mc].

We fix basic terminology and notation in Section 2 and introduce the hyperwebs of quadrics in Section 3. We prove that, for any $r \geq 2$ and for any $n \geq r$ such that $n \equiv r \pmod{2}$, the moduli space $I_{n,r}$ is nonempty and is realized as a free quotient $MI_{n,r}/(GL(n)/\pm \text{id})$, where $MI_{n,r}$ is a Zariski locally closed subset of an affine space (see Theorem 3.1). Thus $MI_{n,r}$ carries a natural structure of a reduced scheme, and $I_{n,r}$ is an algebraic space. In Section 4 we give an explicit construction of vector bundles from $I_{n,r}$ for the above values of n and r and introduce a component $I_{n,r}^*$ of $I_{n,r}$ characterized by a certain open condition (*), see Definition 4.6. In Section 5 we prove Theorem 5.3 on the irreducibility of $I_{n,r}^*$, the main result of this paper.

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2. NOTATION AND CONVENTIONS

In many respects, we follow the exposition of [T], and we stick to the notation introduced therein. The base field \mathbf{k} is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If \mathcal{F} is a sheaf of \mathcal{O}_X -modules on an algebraic variety or a scheme X , then $n\mathcal{F}$ denotes a direct sum of n copies of \mathcal{F} , $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of \mathcal{F} , $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and \mathcal{F}^\vee denotes the dual of \mathcal{F} , that is, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf \mathcal{F} . For any morphism of \mathcal{O}_X -sheaves $f : \mathcal{F} \rightarrow \mathcal{F}'$ and any \mathbf{k} -vector space U (respectively, for any homomorphism $f : U \rightarrow U'$ of \mathbf{k} -vector spaces) we will denote, for short, by the same letter f the induced morphism of sheaves $\text{id} \otimes f : U \otimes \mathcal{F} \rightarrow U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes \text{id} : U \otimes \mathcal{F} \rightarrow U' \otimes \mathcal{F}$).

We fix an integer $n \geq 1$ and denote by H_n a fixed n -dimensional vector space over \mathbf{k} . Throughout the paper, V will be a fixed vector space of dimension 4 over \mathbf{k} , and we set $\mathbb{P}^3 := P(V)$. We reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \rightarrow 0$. For any \mathbf{k} -vector spaces U and W and any vector $\phi \in \text{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \text{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a linear map $\phi : U \otimes V \rightarrow W \otimes V^\vee$ or, equivalently, as a map $\sharp\phi : U \rightarrow W \otimes \wedge^2 V^\vee$, we will denote by $\tilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \rightarrow \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \rightarrow 0$ obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer $m \geq 1$, we denote by \mathbf{S}_m (resp. Σ_{m+1}) the vector space $S^2 H_m^\vee \otimes \wedge^2 V^\vee$ (resp. $\text{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$). Abusing notation, we will denote by the same symbol a \mathbf{k} -vector space, say U , and the associated affine space $\mathbf{V}(U^\vee) = \text{Spec}(\text{Sym}^* U^\vee)$.

All the schemes considered in the paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme \mathcal{X} we mean any closed point of some dense open subset of \mathcal{X} . An irreducible scheme is called generically reduced if it is reduced at a general point.

3. GENERALITIES ON SYMPLECTIC INSTANTONS AND DEFINITION OF $MI_{n,r}$

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [T, Section 3].

For a given symplectic (n, r) -instanton E , the first condition (3) yields $h^0(E(-i)) = 0, i \geq 0$, which, together with the exact triple $0 \rightarrow E(-j-1) \rightarrow E(-j) \rightarrow E(-j)|_{\mathbb{P}^2} \rightarrow 0$ for $j = 0$ and (3), implies that $h^0(E(-1)|_{\mathbb{P}^2}) = 0$, hence also $h^0(E(-i)|_{\mathbb{P}^2}) = 0, i \geq 1$. The last equality for $i = 2$, together with (3) and the above triple for $j = 2$, gives $h^1(E(-3)) = 0$, hence also $h^1(E(-4)) = 0$. Then, from Serre duality and (4), we deduce:

$$(5) \quad h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, \quad i \neq 1, \quad h^i(E(-2)) = 0, \quad i \geq 0.$$

By Riemann-Roch and (3), (5), we have

$$(6) \quad h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E) = h^2(E(-4)) = 2n - 2r.$$

By tensoring the dual Euler sequence by E we also obtain

$$(7) \quad h^1(E \otimes \Omega_{\mathbb{P}^3}^1) = h^2(E \otimes \Omega_{\mathbb{P}^3}^2) = 2n + 2r,$$

Consider a triple (E, f, ϕ) where E is a (n, r) -instanton, $f : H_n \xrightarrow{\sim} H^2(E(-3))$ an isomorphism and $\phi : E \xrightarrow{\sim} E^\vee$ a symplectic structure on E . Two triples (E, f, ϕ) and (E', f', ϕ') are called equivalent if there is an isomorphism $g : E \xrightarrow{\sim} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $\phi = g^\vee \circ \phi' \circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\sim} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, \phi]$ the equivalence class of a triple (E, f, ϕ) . It follows from this definition that the set $F_{[E]}$ of all equivalence classes $[E, f, \phi]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm \text{id}\}$.

Each class $[E, f, \phi]$ defines a point

$$(8) \quad A = A([E, f, \phi]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee$$

in the following way. Consider the exact sequences

$$(9) \quad \begin{aligned} 0 &\rightarrow \Omega_{\mathbb{P}^3}^1 \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0, \\ 0 &\rightarrow \Omega_{\mathbb{P}^3}^2 \rightarrow \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \Omega_{\mathbb{P}^3}^1 \rightarrow 0, \\ 0 &\rightarrow \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega_{\mathbb{P}^3}^2 \rightarrow 0, \end{aligned}$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and taking into account (3), (5)-(7), we obtain the vanishing

$$(10) \quad h^0(E \otimes \Omega_{\mathbb{P}^3}) = h^3(E \otimes \Omega_{\mathbb{P}^3}^2) = h^2(E \otimes \Omega_{\mathbb{P}^3}) = 0$$

and the diagram with exact rows

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^2(E(-4)) \otimes \wedge^4 V^\vee & \longrightarrow & H^2(E(-3)) \otimes \wedge^3 V^\vee & \xrightarrow{i_2} & H^2(E \otimes \Omega_{\mathbb{P}^3}^2) \longrightarrow 0 \\ & & & & \downarrow A' & & \cong \uparrow \partial \\ 0 & \longleftarrow & H^1(E) & \longleftarrow & H^1(E(-1)) \otimes V^\vee & \xleftarrow{i_1} & H^1(E \otimes \Omega_{\mathbb{P}^3}) \longleftarrow 0, \end{array}$$

where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (9) yields the canonical isomorphism $\omega_{\mathbb{P}^3} \xrightarrow{\sim} \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : \mathbf{k} \xrightarrow{\sim} \wedge^4 V^\vee$ we have the isomorphisms $\tilde{\tau} : V \xrightarrow{\sim} \wedge^3 V^\vee$ and $\hat{\tau} : \omega_{\mathbb{P}^3} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^3}(-4)$. We define A in (8) as the composition

$$(12) \quad A : H_n \otimes V \xrightarrow{\tilde{\tau}} H_n \otimes \wedge^3 V^\vee \xrightarrow{f} H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes V^\vee \xrightarrow{\phi} \tau$$

$$\xrightarrow[\cong]{\phi} H^1(E^\vee(-1)) \otimes V^\vee \xrightarrow[\cong]{SD} H^2(E(1) \otimes \omega_{\mathbb{P}^3})^\vee \otimes V^\vee \xrightarrow[\cong]{\hat{\tau}} H^2(E(-3))^\vee \otimes V^\vee \xrightarrow[\cong]{f^\vee} H_n^\vee \otimes V^\vee,$$

where SD is the Serre duality isomorphism. One can verify that A is a skew symmetric map which depends only on the class $[E, f, \phi]$, but does not depend on the choice of τ , and that $A \in \wedge^2(H_n^\vee \otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2 H_n^\vee \otimes \wedge^2 V^\vee$ of the canonical decomposition

$$(13) \quad \wedge^2(H_n^\vee \otimes V^\vee) = S^2 H_n^\vee \otimes \wedge^2 V^\vee \oplus \wedge^2 H_n^\vee \otimes S^2 V^\vee.$$

Here \mathbf{S}_n is the space of hyperwebs of quadrics in H_n . For this reason we call A the (n, r) -instanton hyperweb of quadrics corresponding to the data $[E, f, \phi]$.

Denote $W_A := H_n \otimes V / \ker A$. Using the above chain of isomorphisms we can rewrite the diagram (11) as

$$(14) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker A & \longrightarrow & H_n \otimes V & \xrightarrow{c_A} & W_A \longrightarrow 0 \\ & & & & \downarrow A & & \cong \downarrow q_A \\ 0 & \longleftarrow & \ker A^\vee & \longleftarrow & H_n^\vee \otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee \longleftarrow 0. \end{array}$$

In view of (7), $\dim W_A = 2n + 2r$ and $q_A : W_A \xrightarrow{\cong} W_A^\vee$ is a skew-symmetric isomorphism. An important property of $A = A([E, f, \phi])$ is that the induced morphism of sheaves

$$(15) \quad a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A^\vee} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{ev} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

is surjective and the composition $H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q_A} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is zero. Applying Beilinson spectral sequence [Bei] to $E(-1)$, we see that $E \simeq \ker(a_A^\vee \circ q_A) / \text{Im } a_A$. Thus A defines a monad

$$(16) \quad \mathcal{M}_A : 0 \rightarrow H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

whose cohomology sheaf

$$(17) \quad E_{2r}(A) := \ker(a_A^\vee \circ q_A) / \text{Im } a_A.$$

is isomorphic to E . Twisting \mathcal{M}_A by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (17), we obtain the isomorphism $f : H_n \xrightarrow{\cong} H^2(E(-3))$. Furthermore, the fact that q_A is symplectic implies that there is a canonical isomorphism of \mathcal{M}_A with its dual which induces the symplectic isomorphism $\phi : E \xrightarrow{\cong} E^\vee$. Thus, the data $[E, f, \phi]$ are recovered from A . This leads to the following description of the moduli space $I_{n,r}$. Consider the set of (n, r) -instanton hyperwebs of quadrics

$$(18) \quad MI_{n,r} := \left\{ A \in \mathbf{S}_n \left| \begin{array}{l} \text{(i) } rk(A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee) = 2n + 2r, \\ \text{(ii) the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ defined} \\ \text{by } A \text{ in (15) is surjective,} \\ \text{(iii) } h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee \circ q_A) / \text{Im } a_A \\ \text{and } q_A : W_A \xrightarrow{\cong} W_A^\vee \text{ is a symplectic isomorphism} \\ \text{associated to } A \text{ by (14).} \end{array} \right. \right\}$$

It is a locally closed subscheme of the affine space \mathbf{S}_n .

Theorem 3.1. *The natural morphism*

$$(19) \quad \pi_{n,r} : MI_{n,r} \rightarrow I_{n,r}, \quad A \mapsto [E_{2r}(A)],$$

is a principal $GL(H_n)/\{\pm \text{id}\}$ -bundle in the étale topology. Hence $I_{n,r}$ is a quotient stack $MI_{n,r}/(GL(H_n)/\{\pm \text{id}\})$, making it an algebraic space.

Proof. See [T, Section 3]. \square

Each fibre $F_{[E]} = \pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_{n,r}$ is a principal homogeneous space of the group $GL(H_n)/\{\pm \text{id}\}$. Hence the irreducibility of $(I_{n,r})_{\text{red}}$ is equivalent to the irreducibility of the scheme $(MI_{n,r})_{\text{red}}$.

We can also state:

Theorem 3.2. *For each $n \geq 1$, the space $MI_{n,r}$ of (n, r) -instanton hyperwebs of quadrics is a locally closed subscheme of the vector space \mathbf{S}_n given locally at any point $A \in MI_{n,r}$ by*

$$(20) \quad \binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1)$$

equations obtained as the rank condition (i) in (18).

Note that from (20) it follows that

$$(21) \quad \dim_{[A]} MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1)$$

at any point $A \in MI_{n,r}$. Hence,

$$(22) \quad \dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1)$$

at any point $[E] \in I_{n,r}$, since $MI_{n,r} \rightarrow I_{n,r}$ is a principal $GL(H_n)/\{\pm \text{id}\}$ -bundle in the étale topology.

4. EXPLICIT CONSTRUCTION OF SYMPLECTIC INSTANTONS

4.1. Example: symplectic (n, n) -instantons. In this subsection we recall some known facts about symplectic (n, n) -instantons and their relation to usual rank-2 instantons, see [T, Sections 5-6]. We first show that each invertible hyperweb of quadrics $A \in \mathbf{S}_n$ naturally leads to a construction of a symplectic (n, n) -instanton $E_{2n}(A)$ on \mathbb{P}^3 . Given an integer $n \geq 1$, set

$$(23) \quad \mathbf{S}_n^0 := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee \text{ is an invertible map}\}.$$

Then \mathbf{S}_n^0 is a dense open subset of \mathbf{S}_n , and it is easy to see that for any $A \in \mathbf{S}_n^0$ the following conditions are satisfied.

(1) The morphism $\tilde{A} : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ induced by A is a subbundle embedding, and

$$(24) \quad E_{2n}(A) := \text{coker}(\tilde{A})$$

is a symplectic (n, n) -instanton, that is,

$$(25) \quad [E_{2n}(A)] \in I_{n,n}.$$

(2) For all $i \geq 0$,

$$(26) \quad h^i(E_{2n}(A)) = h^i(E_{2n}(A)(-2)) = 0.$$

This follows from the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 (27) \quad 0 & \longrightarrow & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{A}} & H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1) & \xrightarrow{e} & E_{2n}(A) \longrightarrow 0 \\
 & & \downarrow u & & \downarrow v^\vee & & \\
 & & H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow[\simeq]{A} & H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & & \\
 & & \downarrow v & & \downarrow u^\vee & & \\
 0 \rightarrow E_{2n}(A)^\vee & \longrightarrow & H_n \otimes T_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{A}^\vee} & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Thus $\mathbf{S}_n^0 \subset MI_{n,n}$. In fact, the following result is true.

Proposition 4.1. $\mathbf{S}_n^0 = MI_{n,n}$. In particular, $MI_{n,n}$ is irreducible of dimension $3n^2 + 3n$, and hence $I_{n,n}$ is irreducible of dimension $2n^2 + 3n$.

Proof. We have to show that $MI_{n,n} \subset \mathbf{S}_n^0$. Let $A \in MI_{n,n}$. Since $n = r$, by condition (i) from (18) the rank of the hyperweb of quadrics $A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ is $2n + 2r = 4n = \dim H_n^\vee \otimes V^\vee$, hence A is invertible. By (23), this means that $A \in \mathbf{S}_n^0$. \square

Now we proceed to spell out the relation between symplectic (n, n) -instantons and usual rank-2 instantons with second Chern class $2n - 1$. This relation is given at the level of spaces of hyperwebs of quadrics $MI_{n,n}$ and $MI_{2n-1,1}$ interpreted as spaces of monads.

We need some more notation. Let $B \in \mathbf{S}_n^0$. By definition, B is an invertible anti-self-dual map $H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$. Then the inverse

$$(28) \quad B^{-1} : H_n^\vee \otimes V^\vee \rightarrow H_n \otimes V$$

is also anti-self-dual. Consider the vector space $\Sigma_n = H_n^\vee \otimes H_{n-1}^\vee \otimes \wedge^2 V^\vee$. An element $C \in \Sigma_n$ can be viewed as a linear map $C : H_{n-1} \otimes V \rightarrow H_n^\vee \otimes V^\vee$, and its transpose C^\vee as a map $C^\vee : H_n \otimes V \rightarrow H_{n-1}^\vee \otimes V^\vee$. As the composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H_{n-1}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-1} \oplus \wedge^2 H_{n-1}^\vee \otimes S^2 V^\vee$ (cf. (13)). Thus the condition

$$(29) \quad C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-1}$$

makes sense.

Next, consider the upper horizontal triple in (27) with $A = B$. Twisting it by $\mathcal{O}_{\mathbb{P}^3}(1)$ and passing to global sections we obtain the exact triple

$$(30) \quad 0 \rightarrow H_n \xrightarrow{\sharp B} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1)) \rightarrow 0$$

Besides, interpreting $C \in \Sigma_n$ as a map $\sharp C : H_{n-1} \rightarrow H_n^\vee \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-1} \xrightarrow{\sharp C} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$(31) \quad \rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B).$$

Note also that the maps $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ and $C : H_{n-1} \otimes V \rightarrow H_n^\vee \otimes V^\vee$ provide a map $(H_n \oplus H_{n-1}) \otimes V \rightarrow H_n^\vee \otimes V^\vee$, which induces the morphism of sheaves

$$(32) \quad \tau_{B,C} : (H_n \oplus H_{n-1}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Now set

$$(33) \quad X_n := \left\{ (B, C) \in \mathbf{S}_n^0 \times \Sigma_n \mid \begin{array}{l} \text{(i) the condition (29) is satisfied,} \\ \text{(ii) } \rho_{B,C} \text{ in (31) is a subbundle inclusion,} \\ \text{(iii) } \tau_{B,C} \text{ in (32) is a subbundle inclusion.} \end{array} \right\}$$

By definition, X_n is a locally closed subset of $\mathbf{S}_n^0 \times \Sigma_n$. Hence it is naturally endowed with a structure of a reduced scheme.

Now for any direct sum decomposition

$$(34) \quad \xi : H_{2n-1} \xrightarrow{\cong} H_n \oplus H_{n-1},$$

we obtain the corresponding decomposition

$$(35) \quad \tilde{\xi} : \mathbf{S}_{2n-1} \xrightarrow{\cong} \mathbf{S}_n \oplus \Sigma_n \oplus \mathbf{S}_{n-1} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

Thus, considering the set $MI_{2n-1,1}$ of $(2n-1)$ -instanton hyperwebs of quadrics as a subset of \mathbf{S}_{2n-1} , we obtain a natural projection

$$(36) \quad f_n : MI_{2n-1,1} \rightarrow \mathbf{S}_n \oplus \Sigma_n : A \mapsto (A_1(\xi), A_2(\xi)).$$

The following result is proved in [T, Theorems 1.1, 6.1 and Remark 7.6].

Proposition 4.2. *For a general decomposition ξ in (34), there exists a dense open subset $MI_{2n-1,1}(\xi)$ of $MI_{2n-1,1}$ such that the projection f_n in (36) induces an isomorphism or integral schemes*

$$(37) \quad f_n : MI_{2n-1,1}(\xi) \xrightarrow{\cong} X_n : A \mapsto (A_1(\xi), A_2(\xi)).$$

The inverse isomorphism is given by the formula

$$(38) \quad f_n^{-1} : X_n \xrightarrow{\cong} MI_{2n-1,1}(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C).$$

Besides, the projection

$$(39) \quad pr_1 : X_n \rightarrow \mathbf{S}_n^0 : (B, C) \mapsto B$$

is dominant.

It is not hard to check that the morphism $\rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B)$ defined in (31) satisfies the condition ${}^t\rho_{B,C} \circ \rho_{B,C} = 0$, where ${}^t\rho_{B,C}$ is the composition

$${}^t\rho_{B,C} : E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

and ϕ is a symplectic structure on $E_{2n}(B)$ (cf. [T, formulas (71)-(72)]). In other words, we obtain an anti-self-dual monad

$$(40) \quad 0 \rightarrow H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{B,C}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaf

$$(41) \quad E_2(A) = E_2(B, C) := \ker {}^t\rho_{B,C} / \operatorname{im} \rho_{B,C}, \quad A = f_n^{-1}(B, C).$$

Next, by (19) we have the natural projection

$$(42) \quad \pi_{2n-1,1} : MI_{2n-1,1} \rightarrow I_{2n-1,1} : A \mapsto [E_2(A)].$$

We have the following interpretation of the isomorphism (38) on the level of vector bundles:

$$(43) \quad [E_2(B, C)] = \pi_{2n-1,1}(f_n^{-1}(B, C)).$$

Remark 4.3. Note that, according to the definitions (16)-(18) of $MI_{2n-1,1}$ and $MI_{n,n}$, for any $A \in MI_{2n-1,1}$, if $B = A_1(\xi)$ is defined by the direct sum decomposition (35), one has two other anti-self-dual monads

$$(44) \quad \mathcal{M}_A : 0 \rightarrow H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

$$(45) \quad \mathcal{M}_B : 0 \rightarrow H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_B} W_B \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_B^\vee \circ q_B} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaves

$$(46) \quad E_2(A) = \ker(a_A^\vee \circ q_A) / \operatorname{im} a_A, \quad E_{2n}(B) = \ker(a_B^\vee \circ q_B) / \operatorname{im} a_B$$

respectively. Moreover, (40) and (41) provide an isomorphism $w : W_B = H^2(E_2(B) \otimes \Omega_{\mathbb{P}^3}) \xrightarrow{\cong} H^2(E_{2n}(A) \otimes \Omega_{\mathbb{P}^3}) = W_A$. We thus obtain a commutative anti-self-dual diagram relating these monads:

$$(47) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_B} & W_B \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{q_B} & W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_B^\vee} & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\ & & \downarrow i_\xi & & \cong \downarrow w & & w^\vee \uparrow \cong & & \uparrow i_\xi^\vee & & \\ 0 & \longrightarrow & H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_A} & W_A \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{q_A} & W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_A^\vee} & H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0, \end{array}$$

where $i_\xi : H_n \hookrightarrow H_{2n-1}$ is the embedding induced by the decomposition (34). In view of (46) and the canonical isomorphism $H_{2n-1}/i_\xi(H_n) \simeq H_{n-1}$, from this diagram we obtain the monad

$$(48) \quad \mathcal{M}_{A,B} : 0 \rightarrow H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_{A,B}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{a_{A,B}^\vee} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaf

$$(49) \quad E_2(A) = \ker(a_{A,B}^\vee \circ \phi) / \operatorname{im} a_A.$$

We call (48) the *quotient monad* of the monads (44) and (45).

Remark 4.4. Note that, by Proposition 4.2, the set of all diagrams (47) is parametrized by the irreducible variety $I_{2n-1,1}(\xi)$.

4.2. Example: a special family of symplectic (n, r) -instantons. Now assume $n \geq 2$ and, for any integer r , $2 \leq r \leq n-1$, consider an inclusion

$$(50) \quad \tau : H_{2n-r} \hookrightarrow H_{2n-1}$$

such that

$$(51) \quad \tau(H_{2n-r}) \supset i_\xi(H_n).$$

We obtain a hyperweb of quadrics

$$A_\tau \in S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$$

as the image of A under the map $S^2 H_{2n-1}^\vee \otimes \wedge^2 V^\vee \rightarrow S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$ induced by τ . The corresponding monad

$$(52) \quad \mathcal{M}_\tau : 0 \rightarrow H_{2n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_\tau} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_\tau^\vee \circ q_A} H_{2n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

has a rank- $2r$ cohomology bundle

$$(53) \quad E_{2r}(A_\tau) = \ker(a_\tau^\vee \circ q_A) / \operatorname{im} a_\tau.$$

where $a_\tau := a_A \circ \tau$. By construction, $E_{2r}(A_\tau)$ inherits a natural symplectic structure

$$(54) \quad \phi_r : E_{2r}(A_\tau) \xrightarrow{\sim} E_{2r}(A_\tau)^\vee.$$

Besides, in view of (51), the monad (52) can be inserted as a middle row into the diagram (47), extending it to a three-row commutative anti-self-dual diagram. Arguing as in Remark 4.3 we obtain, in addition to the quotient monad (48), two more quotient monads:

$$(55) \quad \mathcal{M}'_\tau : 0 \rightarrow H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H_{n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

$$E_{2r}(A_\tau) = \ker(a'^\vee_\tau \circ \phi) / \operatorname{im} a'_\tau,$$

$$(56) \quad \mathcal{M}''_\tau : 0 \rightarrow H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(B) \xrightarrow{\phi_\tau} E_{2r}(B)^\vee \xrightarrow{a''^\vee_\tau} H_{r-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

$$E_2(A) = \ker(a'^\vee_\tau \circ \phi_\tau) / \operatorname{im} a_A.$$

From (26) and (55) we easily deduce:

$$(57) \quad h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r.$$

By definition, this together with (52)-(54) means that

$$(58) \quad [E_{2r}(A_\tau)] \in I_{2n-r,r}.$$

Remark 4.5. Observe that, in view of (50), the maps τ belong to the set

$$N_{n,r} := \{\tau \in \operatorname{Hom}(H_{2n-r}, H_{2n-1}) \mid \tau \text{ is injective and } \operatorname{im} \tau \supset \operatorname{im} i_\xi\}.$$

When $A \in MI_{2n-1,1}(\xi)$ is fixed, $N_{n,r}$ parametrizes some family of hyperwebs A_τ from $MI_{2n-r,r}$. Since $N_{n,r}$ is a principal $GL(H_{2n-r})$ -bundle over an open subset of the Grassmannian $Gr(n-r, n-1)$, it is irreducible. Thus, by Remark 4.4, the family of the three-row extensions of the diagrams (47) can be parametrized by the irreducible variety $MI_{2n-1,1}(\xi) \times N_{n,r}$. Hence the family $D_{n,r}$ of isomorphism classes of symplectic rank-2r bundles obtained from these diagrams by formula (53) is an irreducible locally closed subset of $I_{2n-r,r}$.

Note that it is a priori not clear whether the closure of $D_{n,r}$ in $I_{2n-r,r}$ is an irreducible component of $I_{2n-r,r}$.

Definition 4.6. Let $2 \leq r \leq n-1$. We say that $A \in MI_{2n-r,r}$ satisfies property (*) if there exists a monomorphism $i : H_n \hookrightarrow H_{2n-r}$ such that the image B of A under the surjection $\mathbf{S}_{2n-r} \twoheadrightarrow \mathbf{S}_n$ induced by i is invertible as a homomorphism $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$.

The property (*) is clearly an open condition on A . Moreover, since $\pi_{2n-r,r} : MI_{2n-r,r} \rightarrow I_{2n-r,r}$ is a principal bundle (Theorem 3.1), if an element $A \in \pi_{2n-r,r}^{-1}([E_{2r}])$ satisfies (*), then any other point $A' \in \pi_{2n-r,r}^{-1}([E_{2r}])$ satisfies (*). We thus say that $[E_{2r}] \in I_{2n-r,r}$ satisfies property (*) if some (hence any) $A \in \pi_{2n-r,r}^{-1}([E_{2r}])$ satisfies property (*). It is obviously an open condition on $[E_{2r}] \in I_{2n-r,r}$.

Remark 4.7. By Proposition 4.2 and using (51), we see that any $[E_{2r}] \in D_{n,r}$, as well as any $A \in f_n^{-1}(D_{n,r})$ satisfies property (*). We define

$$(59) \quad I_{2n-r,r}^* := I_{(1)} \cup \dots \cup I_{(k)},$$

where $I_{(1)}, \dots, I_{(k)}$ are all the irreducible components of $I_{2n-r,r}$ whose general points satisfy property (*). By definition, $D_{n,r} \subset I_{2n-r,r}^*$, hence $I_{2n-r,r}^*$ is nonempty. We also set $MI_{2n-r,r}^* = \pi_{2n-r,r}^{-1}(I_{2n-r,r}^*)$, so that the map $\pi_{2n-r,r} : MI_{2n-r,r}^* \rightarrow I_{2n-r,r}^*$ is a principal bundle with structure group $GL(H_{2n-r})/\{\pm 1\}$.

5. IRREDUCIBILITY OF $I_{2n-r,r}^*$

5.1. A dense open subset $X_{n,r}$ of $MI_{2n-r,r}^*$. Reduction of the irreducibility of $I_{n,r}^*$ to that of $X_{n,r}$. In this section we prove the irreducibility of the component $I_{2n-r,r}^*$ of $I_{2n-r,r}$ defined in (59), see Theorem 5.3. The explicit construction of symplectic instantons in Section 4 gives us a hint to the proof. We proceed along the lines of Subsection 4.1.

Take any $B \in \mathbf{S}_n^0$ and consider it as an invertible anti-self-dual linear map $H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$. Then B^{-1} is also anti-self-dual. Let

$$(60) \quad \Sigma_{n,r} := H_{n-r}^\vee \otimes H_n^\vee \otimes \wedge^2 V^\vee.$$

An element $C \in \Sigma_n$ can be understood as a map $C : H_{n-r} \otimes V \rightarrow H_n^\vee \otimes V^\vee$, and its transpose C^\vee is a map $H_n \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee$. The composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, i.e., it is an element of $\wedge^2(H_{n-r}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-r} \oplus \wedge^2 H_{n-r}^\vee \otimes S^2 V^\vee$ (cf. (13)). We will later impose the condition

$$(61) \quad C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r}.$$

Next, as in (30), we have a well defined epimorphism $\epsilon(B) : H_n^\vee \otimes \wedge^2 V^\vee \rightarrow H^0(E_{2n}(B)(1))$. Besides, interpreting the above element $C \in \Sigma_{n,r}$ as a map ${}^{\sharp}C : H_{n-r} \rightarrow H_n^\vee \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-r} \xrightarrow{{}^{\sharp}C} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$(62) \quad \rho_{B,C} : H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B).$$

Note also that $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ and $C : H_{n-r} \otimes V \rightarrow H_n^\vee \otimes V^\vee$ define a map $(H_n \oplus H_{n-r}) \otimes V \rightarrow H_n^\vee \otimes V^\vee$ which induces the morphism of sheaves

$$(63) \quad \tau_{B,C} : (H_n \oplus H_{n-r}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Now set

$$(64) \quad X_{n,r} := \left\{ (B, C) \in \mathbf{S}_n^0 \times \Sigma_{n,r} \left| \begin{array}{l} \text{(i) the condition (61) is satisfied,} \\ \text{(ii) } \rho_{B,C} \text{ in (62) is a subbundle inclusion,} \\ \text{(iii) } \tau_{B,C} \text{ in (63) is a subbundle inclusion.} \end{array} \right. \right\}$$

By definition, $X_{n,r}$ is a locally closed subset of $\mathbf{S}_n^0 \times \Sigma_{n,r}$. Hence it has a natural structure of reduced scheme.

Now for an arbitrary direct sum decomposition

$$(65) \quad \xi : H_{2n-r} \xrightarrow{\cong} H_n \oplus H_{n-r}$$

we obtain the corresponding decomposition

$$(66) \quad \tilde{\xi} : \mathbf{S}_{2n-r} \xrightarrow{\cong} \mathbf{S}_n \oplus \Sigma_{n,r} \oplus \mathbf{S}_{n-r} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

Thus, considering the set $MI_{2n-r,r}$ of symplectic $(2n-r, r)$ -instanton hyperwebs of quadrics as a subset of \mathbf{S}_{2n-r} , we obtain a natural projection

$$(67) \quad f_{n,r} : MI_{2n-r,r} \rightarrow \mathbf{S}_n \oplus \Sigma_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)).$$

We now prove the following result parallel to Proposition 4.2.

Theorem 5.1. *Let $n \geq 3$ and $2 \leq r \leq n - 1$.*

(i) *For a general decomposition ξ in (65) there is an open dense subset $MI_{2n-r,r}^*(\xi)$ of $MI_{2n-r,r}^*$ and an isomorphism of reduced schemes*

$$(68) \quad f_{n,r} : MI_{2n-r,r}^*(\xi) \xrightarrow{\cong} X_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)),$$

where $A_1(\xi)$ and $A_2(\xi)$ are defined by (66).

(ii) *The inverse isomorphism is given by the formula*

$$(69) \quad f_{n,r}^{-1} : X_{n,r} \xrightarrow{\cong} MI_{2n-r,r}^*(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C),$$

where $\tilde{\xi}$ is defined by (66).

Proof. Set $MI_{2n-r,r}^*(\xi) := \{A \in MI_{2n-r,r}^* \mid A \text{ satisfies property } (*) \text{ for the monomorphism } i : H_n \hookrightarrow H_{2n-r} \text{ defined by } \xi\}$. It follows from Definition 4.6 and Remark 4.7 that, for a general decomposition ξ in (65), $MI_{2n-r,r}^*(\xi)$ is a dense open subset of $MI_{2n-r,r}^*$. Then, for this choice of ξ , the proof of this Theorem essentially mimics the proof of [T, Proposition 6.1] in which we make the substitution $m + 1 \mapsto n$, $m \mapsto n - r$ and change the notation accordingly. \square

The proof of the following theorem will be given in Subsection 5.2.

Theorem 5.2. *$X_{n,r}$ is irreducible of dimension $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$.*

From Theorems 5.1 and 5.2 it follows that $MI_{2n-r,r}^*$ is irreducible of dimension $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$ for any $n \leq 3$ and $2 \leq r \leq n - 1$. Hence $I_{2n-r,r}^*$ is irreducible of dimension $4(2n - r)(r + 1) - r(2r + 1)$ for these values of n and r . Note that the irreducibility of $I_{2n-r,r}^*$ is also true when $r = n$, and in this case $I_{n,n}^*$ coincides with $I_{n,n}$. Substituting $2n - 1 \mapsto n$, we obtain the following main result of the paper.

Theorem 5.3. *For any integer $r \geq 2$ and for any integer $n \geq r$ such that $n \equiv r \pmod{2}$, $I_{n,r}^*$ is an irreducible component of $I_{n,r}$ of dimension $4n(r + 1) - r(2r + 1)$.*

5.2. Proof of the irreducibility of $X_{n,r}$. In this subsection we give the proof of Theorem 5.2. Define

$$(70) \quad \tilde{X}_{n,r} := \{(D, C) \in (\mathbf{S}_n^\vee)^0 \times \Sigma_{n,r} \mid (C^\vee \circ D \circ C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee) \in \mathbf{S}_{n-r}\},$$

a closed subscheme of $(\mathbf{S}_m^\vee)^0 \times \Sigma_{n,r}$ defined by the equations

$$(71) \quad C^\vee \circ D \circ C \in \mathbf{S}_{n-r}.$$

Since the conditions (ii) and (iii) in the definition (33) of $X_{n,r}$ are open and $X_{n,r}$ is nonempty (see Theorem 5.1), the isomorphism

$$\mathbf{S}_n^0 \xrightarrow{\cong} (\mathbf{S}_n^\vee)^0 : B \mapsto B^{-1}$$

implies that $X_{n,r}$ is a nonempty open subset of $(\tilde{X}_{n,r})_{\text{red}}$,

$$(72) \quad \emptyset \neq X_{n,r} \xrightarrow{\text{open}} (\tilde{X}_{n,r})_{\text{red}}.$$

Fix a direct sum decomposition

$$H_n \xrightarrow{\cong} H_{n-r} \oplus H_r.$$

Then any linear map

$$(73) \quad C \in \Sigma_{n,r} = \text{Hom}(H_{n-r}, H_n^\vee \otimes \wedge^2 V^\vee), \quad C : H_{n-r} \otimes V \rightarrow H_n^\vee \otimes V^\vee,$$

can be represented as a map

$$(74) \quad C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee \oplus H_r^\vee \otimes V^\vee,$$

or else as a block matrix

$$(75) \quad C = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

where

$$(76) \quad \phi \in \text{Hom}(H_{n-r}, H_{n-r}^\vee) \otimes \wedge^2 V^\vee = \Phi_{n-r}, \quad \psi \in \Psi_{n,r} := \text{Hom}(H_{n-r}, H_r^\vee) \otimes \wedge^2 V^\vee.$$

Similarly, any $D \in (\mathbf{S}_n^\vee)^0 \subset \mathbf{S}_n^\vee = S^2 H_n \otimes \wedge^2 V \subset \text{Hom}(H_n^\vee \otimes V^\vee, H_n \otimes V)$ can be represented in the form

$$(77) \quad D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix},$$

where

$$(78) \quad D_1 \in \mathbf{S}_{n-r}^\vee \subset \text{Hom}(H_{n-r}^\vee \otimes V^\vee, H_{n-r} \otimes V),$$

$$\lambda \in \mathbf{L}_{n,r} := \text{Hom}(H_r^\vee, H_{n-r}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_r := S^2 H_r \otimes \wedge^2 V.$$

By (75) and (77) the composition

$$C^\vee \circ D \circ C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee \quad (C^\vee \circ D \circ C \in \wedge^2(H_{n-r}^\vee \otimes V^\vee))$$

can be written in the form

$$(79) \quad C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi.$$

By (75)-(78) we have

$$\mathbf{S}_n^\vee \times \Sigma_{n,r} = \mathbf{S}_{n-r}^\vee \times \Phi_{n-r} \times \Psi_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_r,$$

and there are well defined morphisms

$$\tilde{p} : \tilde{X}_{n,r} \rightarrow \mathbf{L}_{n,r} \times \mathbf{M}_r : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$p := \tilde{p}|_{\overline{X}_{n,r}} : \overline{X}_{n,r} \rightarrow \mathbf{L}_{n,r} \oplus \mathbf{M}_r,$$

where $\overline{X}_{n,r}$ is the closure of $X_{n,r}$ in $(\mathbf{S}_n^\vee)^0 \times \Sigma_{n,r}$. We now invoke the following result from [T]:

Proposition 5.4. *Let $n \geq 2$. Then for any $D \in (\mathbf{S}_n^\vee)^0$ and for a general choice of the decomposition $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$, the block D_1 of D in (77) is nondegenerate.*

Proof. See [T, Proposition 7.3]. By repeatedly applying this proposition r times, we can find a decomposition $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$ such that $D_1 : H_{n-r}^\vee \otimes V^\vee \rightarrow H_{n-r} \otimes V$ in (77) is nondegenerate, i.e., $D_1 \in (\mathbf{S}_{n-r}^\vee)^0$. \square

Let \mathcal{X} be any irreducible component of $X_{n,r}$ and let $\overline{\mathcal{X}}$ be its closure in $\overline{X}_{n,r}$. Fix a point $z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X}$ not lying in the components of $X_{n,r}$ different from \mathcal{X} . Consider the morphism

$$(80) \quad f : \mathbb{A}^1 \rightarrow \overline{\mathcal{X}} : t \mapsto (D_1, t^2 \phi, t \psi, t \lambda, t^2 \mu), \quad f(1) = z,$$

which is well defined by (79). By definition, the point $f(0) = (D_1, 0, 0, 0, 0)$ lies in the fibre $p^{-1}(0, 0)$. Hence, $p^{-1}(0, 0) \cap \overline{\mathcal{X}} \neq \emptyset$. In other words,

$$(81) \quad \rho^{-1}(0, 0) \neq \emptyset, \quad \text{where } \rho := p|_{\overline{\mathcal{X}}}.$$

Now, it follows from (79) and the definition of $\tilde{X}_{n,r}$ that

$$(82) \quad \tilde{p}^{-1}(0, 0) = \{(D_1, \phi, \psi) \in (\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r} \times \Psi_{n,r} \mid \phi^\vee \circ D_1 \circ \phi \in \mathbf{S}_{n-r}\}.$$

Consider the set

$$Z_{n-r} = \{(D, \phi) \in (\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r} \mid \phi^\vee \circ D \circ \phi \in \mathbf{S}_{n-r}\}.$$

It carries a natural scheme structure, where it is a closed subscheme of $(\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r}$. Comparing the definition of Z_{n-r} with (82) we see that there are scheme-theoretic inclusions of schemes

$$(83) \quad \rho^{-1}(0, 0) \subset p^{-1}(0, 0) \subset \tilde{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.$$

By [T, Theorem 7.2], Z_{n-r} is an integral scheme of dimension $4(n-r)(n-r+2)$. This together with (83) implies that

$$(84) \quad \dim \rho^{-1}(0, 0) \leq \dim p^{-1}(0, 0) \leq \dim Z_{n-r} + \dim \Psi_{n,r} = 4(n-r)(n-r+2) + 6r(n-r) = (n-r)(4n+2r+8).$$

Hence in view of (81)

$$(85) \quad \dim \overline{\mathcal{X}} \leq \dim \rho^{-1}(0, 0) + \dim \mathbf{L}_{n,r} + \dim \mathbf{M}_r \leq (n-r)(4n+2r+8) + 6r(n-r) + 3r(r+1) = (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1).$$

On the other hand, formula (21), with $2n-r$ substituted for n , and Theorem 5.1(ii) show that, for any point $x \in \mathcal{X}$ such that $A := f_{n,r}^{-1}(x) \in MI_{2n-r,r}^0(\xi)$,

$$(86) \quad (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1) \leq \dim_A MI_{2n-r,r}^0(\xi) = \dim \overline{\mathcal{X}}.$$

Comparing (85) with (86), we see that all the inequalities in (84)-(86) are equalities. In particular,

$$(87) \quad \dim \rho^{-1}(0, 0) = \dim(Z_{n-r} \times \Psi_{n,r}) = \dim \overline{\mathcal{X}} - \dim(\mathbf{L}_{n,r} \times \mathbf{M}_r).$$

Since by Theorem [T, Theorem 7.2] the scheme Z_{n-r} is integral and so $Z_{n-r} \times \Psi_{n,r}$ is integral as well, (83) and (87) yield the equalities of integral schemes

$$(88) \quad \rho^{-1}(0, 0) = p^{-1}(0, 0) = \tilde{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.$$

Now we invoke one auxiliary result from [T].

Lemma 5.5. *Let $f : X \rightarrow Y$ be a morphism of reduced schemes, where Y is a smooth integral scheme. Assume that there exists a closed point $y \in Y$ such that for any irreducible component X' of X the following conditions are satisfied:*

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$,

(b) *the scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.*

Then

(i) *there exists an open subset U of Y containing the point y such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is flat, and*

(ii) *X is integral.*

Proof. See [T, Lemma 7.4]. □

Applying assertions (i)-(ii) of this lemma to $X = X_{n,r}$, $X' = \mathcal{X}$, $Y = \mathbf{L}_{n,r} \times \mathbf{M}_r$, $y = (0, 0)$, $f = p$, and using (87) and (88), we obtain that $X_{n,r}$ is integral of dimension $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$. Theorem 5.2 is proved.

REFERENCES

- [A] **Atiyah, M. F.**, *Geometry of Yang-Mills fields*, Scuola Normale Superiore, Pisa, 1979, 99 pp.
- [ADHM] **Atiyah, M. F., Drinfeld, V. G., Hitchin, N. J., Manin, Yu. I.**, *Construction of instantons*, Phys. Lett. **A 65** (1978), 185-187.
- [AW] **Atiyah, M. F., and Ward, R. S.**, *Instantons and algebraic geometry*, Comm. Math. Phys. **55** (1977), 117-124.
- [B] **Barth, W.**, Lectures on mathematical instanton bundles, in: Gauge Theories: Fundamental Interactions and Rigorous Results, P. Dita, V. Georgescu, and R. Purice, eds., Birkhäuser, Boston, 1982, pp. 177-206.
- [BH] **Barth, W., Hulek K.**, *Monads and moduli of vector bundles*, Manuscripta Math. **25** (1978), 323-347.
- [Bei] **Beilinson, A.**, *Coherent sheaves on \mathbb{P}^n and problems in linear algebra* (Russian) Funktsional. Anal. i Prilozhen. **12** (1978), 68-69.
- [H] **Horrocks, G.**, *Vector bundles on the punctured spectrum of a local ring*, Proc. Lond. Math. Soc. **14** (1964), 684-713.
- [JV] **Jardim, M., Verbitsky, M.**, *Trihypercähler reduction and instanton bundles on \mathbb{CP}^3* , arXiv:1103.4431.
- [Mc] **McCarthy, P. J.**, *Rational parametrisation of normalised Stiefel manifolds and explicit non-'t Hooft solutions of the ADHM instanton matrix equations for $Sp(n)$* , Lett. Math. Phys. **5** (1981) 255-261.
- [T] **Tikhomirov, A. S.**, *Moduli of mathematical instanton vector bundles with odd c_2 on projective space*, Preprint arXiv:1101.3016.
- [Tju1] **Tyurin, A. N.**, *On the superposition of mathematical instantons II*, In: Arithmetic and Geometry, Progress in Mathematics 36, Birkhäuser 1983.
- [Tju2] **Tyurin, A. N.**, *The structure of the variety of pairs of commuting pencils of symmetric matrices*, Math. USSR Izvestiya, **20(2)** (1983), 391-410.

U. BRUZZO: SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI, VIA BONOMEA 265, 34136 TRIESTE, ITALIA
AND ISTITUTO NAZIONALE DI FISICA NUCLEARE, SEZIONE DI TRIESTE

E-mail address: `bruzzo@sissa.it`

D. MARKUSHEVICH: MATHÉMATIQUES - BÂT.M2, UNIVERSITÉ LILLE 1, F-59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

E-mail address: `markushe@math.univ-lille1.fr`

A.S. TIKHOMIROV: DEPARTMENT OF MATHEMATICS, YAROSLAVL STATE PEDAGOGICAL UNIVERSITY, RESPUBLIKAN-SKAYA STR. 108, 150 000 YAROSLAVL, RUSSIA

E-mail address: `astikhomirov@mail.ru`